

where $\{Q\}$ = sum of loads up to the particular excavation sequence minus loads in the elements that were just excavated.

Embankment construction

Simulation of embankment construction is relatively straightforward and involves successive addition of elements with load for each sequence evaluated from

Support systems

One-dimensional bar elements with linear and quadratic variation of displacement are used to simulate tie-backs, braces and anchors. Linings in tunnels and cavities are idealized by using the eight-node solid element.

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THE POOR BENDING RESPONSE OF THE FOUR-NODE PLANE STRESS QUADRILATERAL

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SUMMARY

Many plane stress finite elements which can exactly represent rigid-body and constant-strain modes are too stiff in their response to the simple flexural action of a beam. This problem has received considerable interest. In this paper we explore a new interpretation of the problem and show that the poor bending response of the original 4-node plane stress quadrilateral can be quantitatively predicted by an error model.

INTRODUCTION

Many plane stress finite elements are too stiff in their response to the simple flexural action of a beam, e.g. the simple 4-node quadrilateral element with eight nodal freedoms, u_1-u_4 , v_1-v_4 (Figure 1a). Explanations based on its inability to bend into the simple curved shape required for flexural modes (Figure 1b) helped in the understanding of the remarkable improvement obtained by adding non-conforming modes.^{1,2} Alternatively, a reduced integration of the shear strain energy,^{3,4} modified shear strain interpolations,^{4,5} non-conforming displacement formulation⁶ and an artificial softening procedure⁷ provided ways to improve the bending response of the plane stress element. Here, we seek a new interpretation of the problem, derive a quantitative error estimate and show why the various modifications¹⁻⁷ are effective.

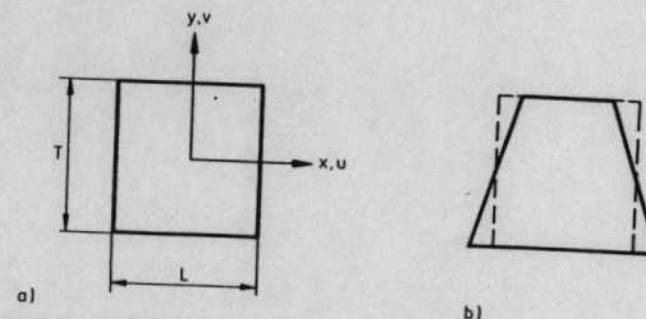


Figure 1(a). Four-node quadrilateral plane stress element, (b) suggested geometrical description of its response to a bending mode

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FIELD CONSISTENCY INTERPRETATION

It was believed that the completeness of the polynomial fields used as shape functions and the continuity of these functions and their derivatives, where required, across inter-element boundaries ensured the convergence of finite elements. However, elements satisfying these principles still behaved very poorly in some practical cases.

Clearly, it was necessary to differentiate the kind of errors defined by the requirements of the mathematical precision of the problem statement from the kind of errors that emerged from the inability of the element to model a typical response of an engineering structure. On the basis of a classical order of error analysis, convergence can be shown to improve as the mesh is refined—we shall refer to errors which are quickly removed by mesh refinement as errors of the first kind. It has also become clear, recently, that the shape function definitions and subsequent discretization by the integration of functionals over the element volume can introduce constraints on the continuous media being modelled. While some of these constraints may be truly required by the actual physical model, other spurious constraints may emerge, as the shape functions do not 'consistently' model the field being studied. These cause errors of the second kind. They result in rapid deterioration of accuracy due to multiplication by parameters arising from the structural modelling. When these parameters take very large values in a penalty limiting sense, the solution 'locks' to zero values.

Walz *et al.*⁸ recognized the existence of a kind of discretization error which did not vanish rapidly as element size was reduced. Prathap and Bhashyam⁹ classified errors arising from 'spurious constraints' in a shear flexible beam element into this second class. This interpretation helped to understand shear locking in plates¹⁰ and membrane locking in curved beams and thin shells.¹¹ This interpretation is now extended to understand the poor behaviour of simple plane stress elements in bending.

A 'field consistency' interpretation¹² recognizes that the independent bilinear interpolations for u and v displacements in a plane stress field cannot model the shear strain field consistently in beam flexure. These inconsistencies emerge as severe constraints, at limiting situations of physical behaviour at large slenderness ratios and cause very large errors. We shall demonstrate this analytically for the 4-node element in its rectangular form.

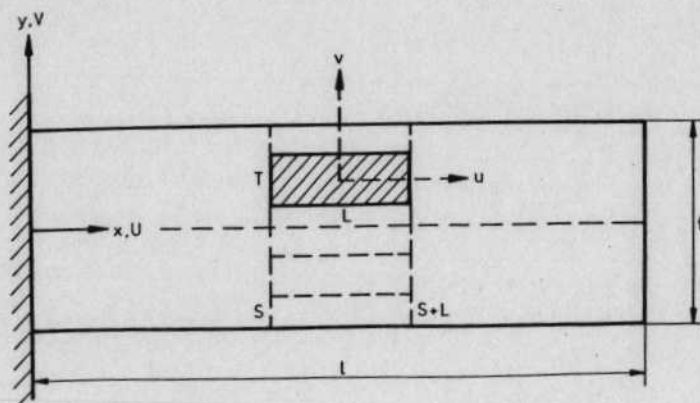


Figure 2. Rectangular beam and rectangular plane stress element

ANALYTICAL ERROR MODEL FOR BEAM FLEXURE

Figure 2 describes a rectangular beam of length l and depth t , modelled by plane stress elements of dimension L by T . The strain energy of the entire beam, in terms of the displacement fields U and V , is

$$\pi = \int_0^l \int_0^t \left[\frac{Eb}{2(1-\nu^2)} (U_{,x}^2 + V_{,y}^2 + 2\nu U_{,x} V_{,y}) + \frac{Gb}{2} (U_{,y} + V_{,x})^2 \right] dx dy \quad (1)$$

E and G are the Young's and shear moduli and ν is the Poisson's ratio, and

$$\pi = \pi(E) + \pi(G) \quad (2)$$

where $\pi(E)$ is the strain energy contribution from the normal strains and $\pi(G)$ comes from the shear strains.

An element of length L and depth T is isolated. The expression for the element strain energy π_e remains the same as equation (1), with the integration limits changing to $x=0$ to L and $y=0$ to T . Consider a 4-node element, with the field functions defined as

$$\begin{aligned} u &= a_0 + a_1 x + a_2 y + a_3 xy \\ v &= b_0 + b_1 x + b_2 y + b_3 xy \end{aligned} \quad (3)$$

In the field consistency approach, the critical strain field must show the correct constraining conditions at limiting physical situations. For a very slender beam, the shear strain must vanish in the Kirchhoff or Euler-Bernoulli sense. Let us see how equations (3) will contribute to the shear related strain within the element:

$$\begin{aligned} \pi_e(G) &= \int_0^L \int_0^T \frac{Gb}{2} (a_2 + a_3 x + b_1 + b_3 y)^2 dx dy \\ &= \frac{Gb}{2} LT \left[\left(a_2 + a_3 \frac{L}{2} + b_1 + b_3 \frac{T}{2} \right)^2 + a_3 \frac{L^2}{12} + b_3 \frac{T^2}{12} \right] \end{aligned} \quad (4)$$

For the purely flexural action one expects in very thin beams, the shear energy vanishes as constraints are produced by $G \rightarrow \infty$. From equation (4), these constraints are

$$a_2 + a_3 \frac{L}{2} + b_1 + b_3 \frac{T}{2} = 0 \quad (5a)$$

$$a_3 = 0 \quad (5b)$$

$$b_3 = 0 \quad (5c)$$

or alternatively

$$(u_{,y} + v_{,x})_0 = 0 \quad (6a)$$

$$(u_{,xy})_0 = 0 \quad (6b)$$

$$(v_{,xy})_0 = 0 \quad (6c)$$

where the subscript zero denotes values at the centroid of the element. The constraint represented by equations (5a) or (6a) has terms from both interpolation functions for u and v and can reproduce a true state of zero shear strain in the limiting case. However, constraints represented by equations (5b) and (5c) contain, in each case, a term from one field function alone. Thus, in the limit, these cause unnecessary restrictions on the behaviour of the field functions for u and

v. We see below how an examination of these constraints can produce an analytical prediction for the error in such finite element representation.

From the two constraints (5b) or (6b) and (5c) or (6c), it is necessary to know how severe the contributions of each are to errors of the second kind. Equation (4), shows that the first constraint is multiplied by an L^2 term, while the second constraint is multiplied by a T^2 term. The energy contributions from these two terms are in an $(L/T)^2$ ratio. Since one would favour modelling a beam by geometrically similar elements, i.e. $L/T = l/t$, for a very slender beam, we can see that the spurious energy term from $a_3 \rightarrow 0$ will dominate the contributions to errors of the second kind.

Next, we examine how this spurious energy term will compare with the normal strain energy contribution $\pi_e(E)$. For this, we start with a simple model of a beam undergoing flexure. In engineering theory, the energy in beam flexure is

$$\pi = \frac{1}{2} \int_0^L \frac{Ebt^3}{12} w_{,xx}^2 dx \quad (7)$$

where $w(x)$ is the transverse deflection of the neutral axis.

A simple theory of elasticity solution approximates this two dimensionally as

$$\begin{aligned} U(x, y) &= -yW_{,x} \\ V(x, y) &= W \end{aligned} \quad (8)$$

From this, we have

$$U_{,y} + V_{,x} = 0 \quad (9a)$$

$$U_{,xy} = -W_{,xx} \quad (9b)$$

$$V_{,xy} = 0 \quad (9c)$$

Comparing equations (9a)–(9c) with equations (6a)–(6b), we notice that:

1. The shear strain = 0 condition represented by equation (5a) or (6a) is realized over the entire domain of the beam.
2. The condition (5b) or (6b) is unrealistic as it would enforce $W_{,xx} = 0$ within each element and hence over the whole domain. Clearly, this is the offending 'spurious' constraint in a thin beam situation.
3. The condition (5c) or (6c) is realistic, as it is true over the whole domain for this thin beam theory.

In what follows, we see how the unrealistic constraint represented by equation (5b) will cause the 'parasitic shear locking'. For this we must derive the expression for the strain energy of N elements located as shown in Figure 2, between $x = s$ and $x = s+L$.

We assume that the true constraint represented by equation (5a) is correctly enforced and does not contribute to the shear strain energy in the very thin limit. We also assume that the contribution from the third constraint is vanishingly small, as it is a very small fraction of the second constraint, equation (5b), and/or because it is also a realistic constraint, after the prediction of equation (9c). The total strain energy predicted after discretization (following References 9–11), is

$$\pi'_{s,s+L} = \int_s^{s+L} \sum_1^N \left[\int_0^T \frac{Eb}{2(1-\nu^2)} (U_{,x}^2 + V_{,y}^2 + 2\nu U_{,x} V_{,y}) dy + \frac{GbLT^2}{2 \cdot 12} (U_{,xy})_0^2 \right] dx \quad (10)$$

From equation (9b), for all the N elements, we can replace $(U_{,xy})_0$ by $-W_{,xx}$, and as this is

independent of y , we introduce the following summation:

$$\frac{GbL^2}{24} \sum_1^N T(U_{,xy})_0^2 = \frac{GbL^2}{24} tW_{,xx}^2$$

Thus

$$\pi'_{s,s+L} = \int_s^{s+L} \sum_1^N \int_0^T \frac{Eb}{2(1-\nu^2)} (U_{,x}^2 + V_{,y}^2 + 2\nu U_{,x} V_{,y}) dy dx + \int_s^{s+L} \frac{GbL^2}{24} tW_{,xx}^2 dx \quad (11)$$

It is well known¹³ that when only one element is used through the thickness, the normal strain energy is altered by a factor $(1-\nu^2)$. It is possible to see how this emerges, in this analytical model. From equation (8),

$$\begin{aligned} U_{,x} &= -yW_{,xx} \\ V_{,y} &= 0 \end{aligned} \quad (12)$$

Thus, where only one plane stress element is used through the thickness, the condition $V_{,y} = 0$ is enforced, and the altered strain energy, after introducing the terms from equations (12) and integrating through the thickness, becomes, for the entire idealized beam,

$$\pi'_1 = \pi'_{N-1} = \frac{1}{2} \int_0^L \frac{Ebt^3}{12(1-\nu^2)} W_{,xx}^2 dx + \frac{1}{2} \int_0^L \frac{Gbt}{2} L^2 W_{,xx}^2 dx \quad (13)$$

Also, from the theory of elasticity models of a beam with shear flexibility taken into account, we have $V_{,y} = -\nu U_{,x}$. This condition, it is expected, can be correctly modelled only when sufficiently large numbers of elements are used through the thickness. The altered strain energy of this idealized beam is then, from equation (11),

$$\pi'_{\infty} = \pi'_{n=\infty} = \frac{1}{2} \int_0^L \frac{Ebt^3}{12} W_{,xx}^2 dx + \frac{1}{2} \int_0^L \frac{Gbt}{2} L^2 W_{,xx}^2 dx \quad (14)$$

We now compare the original continuum theory prediction for the strain energy stored in a thin beam undergoing flexure, with the continuum equivalents of the idealized beams. Grouping them in one place, we have

$$\pi = \frac{1}{2} \int_0^L \frac{Ebt^3}{12} W_{,xx}^2 dx \quad (15a)$$

$$\pi'_1 = \frac{1}{2} \int_0^L \frac{Ebt^3}{12} \left(\frac{1}{1-\nu^2} + \frac{GL^2}{Et^2} \right) W_{,xx}^2 dx \quad (15b)$$

$$\pi'_{\infty} = \frac{1}{2} \int_0^L \frac{Ebt^3}{12} \left(1 + \frac{GL^2}{Et^2} \right) W_{,xx}^2 dx \quad (15c)$$

In the discretized models, therefore, a spurious shear-related energy originating from the inconsistent constraint represented by equation (5b) or equation (6b) has appeared. Due to this spurious constraint, the original rigidity $I = bt^3/12$ has been altered by the modelling in the following fashion:

$$\frac{I'_1}{I} = \left(\frac{1}{1-\nu^2} + \frac{GL^2}{Et^2} \right) \quad (16a)$$

$$\frac{I'_{\infty}}{I} = \left(1 + \frac{GL^2}{Et^2} \right) \quad (16b)$$

Note now that the Poisson's effect present in equation (16a) is of minor nature. It is relieved as the idealization through the thickness is increased. It also does not constitute an error of the second kind, as it is not exaggerated by changing any structural parameter. However, the GL^2/Et^2 term represents the parasitic-shear locking. Interestingly, it is a function of the ratio L/t and not the element aspect ratio L/T as one would have expected. This explains why improvement through the thickness does not relieve the parasitic shear locking. It is also interesting to note that the factor required⁷ to achieve artificial softening of the same plane stress element is identical to the factor given on the right-hand side of equation (16a).

The next section shows how typical numerical results can be related to the analytical error models derived in this section. We shall also see why a simple application of the softening factor to the stiffness matrix causes the bending stresses to be overestimated by an additional factor $(1 - \nu^2)$.

NUMERICAL EXAMPLES

In a previous exercise,¹⁴ a simple error norm, the additional stiffening parameter showed how errors of the second kind can be blown out of proportion by a large variation in some structural parameter. The altered rigidities given by equations (16a) and (16b) suggest how this norm can be defined for this element. A reference deflection, say the maximum deflection at the tip of the beam, is compared for the theoretical case and the finite element model. If no errors of the first kind are present (a reasonably good idealization would achieve this), the two deflections would differ in the ratio

$$\frac{W(\text{theory})}{W(\text{fem})} = \frac{I'}{I}$$

Thus, the additional stiffening parameter will be

$$e = \frac{W(\text{theory})}{W(\text{fem})} - 1 = \frac{I'}{I} - 1 \quad (17)$$

Thus, for the two models above represented by equations (16a) and (16b), the additional stiffness parameter would be

$$e_1 = \frac{1}{(1 - \nu^2)} + \frac{GL^2}{Et^2} - 1 \quad (18a)$$

$$e_\infty = \frac{GL^2}{Et^2} \quad (18b)$$

The results given by Cook⁷ are now reinterpreted. A cantilever beam of aspect ratio $l/t = 5$ was studied using only one element through the thickness, and with one five elements, respectively, along the length, under the load configurations shown in Figure 3.

Table I shows the actual vertical deflection at point A taken from the finite element results with the original 4-node plane stress element as presented in Reference 7. The third row gives the values of $e(\text{fem})$, as computed from the numerical results, as

$$e(\text{fem}) = \frac{W(\text{theory})}{W(\text{fem})} - 1$$

and the fourth row in the table gives the value predicted by the analytical model, by evaluating the right-hand side of equation (18a). The excellent agreement proves the accuracy of the analytical

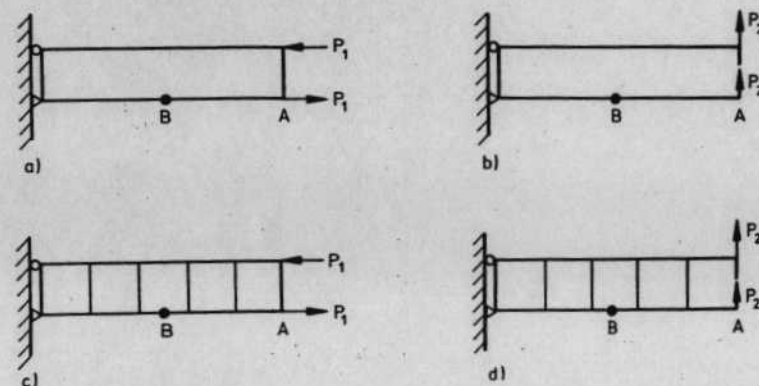


Figure 3. Cantilever beam of aspect ratio $l/t = 5.0$ modelled by one element and five element meshes ($\nu = 0.25$)

model to represent the effect of parasitic shear locking. We also see why the alteration of the flexural mode related stiffness terms by the same softening parameter helped to improve the plane stress elements in its prediction of deflections. The artificial softening had re-altered the rigidity of the finite element model by compensating it with the same factor that emerges from the parasitic shear locking. We can now see why this artificial softening alters the bending stresses by $(1 - \nu^2)$.

Let u, u' and u'' be the fields obtained from the theoretical, the original finite element and the artificially softened element models, using one-element idealizations through the thickness. We have $v_{,y} = -\nu u_{,x}$, $v'_{,y} = 0$ and $v''_{,y} = 0$. Due to locking, from equation (16a)

$$\frac{u}{u'} = \frac{1}{1 - \nu^2} + \frac{GL^2}{Et^2} \quad (19)$$

We also have $u'' = u$. The bending stresses σ_x, σ'_x and σ''_x for the theoretical, original element and softened elements are

$$\sigma_x = \frac{E}{1 - \nu^2} (u_{,x} + \nu v_{,y}) = E u_{,x}$$

$$\sigma'_x = \frac{E}{1 - \nu^2} (u'_{,x} + \nu v'_{,y}) = \frac{E}{1 - \nu^2} u'_{,x}$$

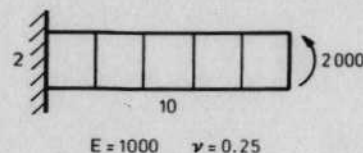
$$\sigma''_x = \frac{E}{1 - \nu^2} (u''_{,x} + \nu v''_{,y}) = \frac{E}{1 - \nu^2} u''_{,x}$$

Table I. Additional stiffness parameter for cases in Figure 3, from deflections at A

Case	Figure (3a)	Figure (3b)	Figure (3c)	Figure (3d)
$W(\text{fem})$	9.0	9.3	68.2	70.2
$W(\text{theory})$	100.0	102.6	100.0	102.6
$e(\text{fem})$	10.1	10.0	0.466	0.462
$e_1(\text{Eqn 18a})$	10.1	10.1	0.466	0.466

Table II. Stress factors for cases in Figure 2 from bending stress at B

Case	Figure (3a)	Figure (3b)	Figure (3c)	Figure (3d)
$\sigma_x(\text{theory})$	3000.0	2250.0	3000.0	2250.0
$\sigma_x(\text{fem})$	289.0	217.0	2182.0	1636.0
$\sigma_x''(\text{fem})$	3200.0	2400.0	3200.0	2400.0
$f'(\text{fem})$	10.38	10.37	1.375	1.375
$f'(\text{Eqn 20a})$	10.38	10.38	1.375	1.375
$f''(\text{fem})$	0.9375	0.9375	0.9375	0.9375
$f''(\text{Eqn 20b})$	0.9375	0.9375	0.9375	0.9375

Figure 4. Additional stiffness parameter for cantilever beam with N elements through the thickness

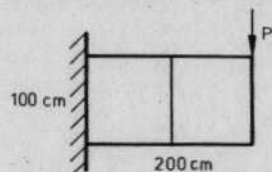
We now define stress factors f' and f'' as

$$f' = \frac{\sigma_x}{\sigma_x'} = (1 - \nu^2) \frac{u_{,x}}{u'_{,x}} = (1 - \nu^2) \left[\frac{1}{1 - \nu^2} + \frac{GL^2}{Et^2} \right] \quad (20a)$$

$$f'' = \frac{\sigma_x}{\sigma_x''} = (1 - \nu^2) \quad (20b)$$

Clearly, from equation (20b), the artificial softening predicts stresses higher by a factor $(1 - \nu^2)$, whereas the original element is in error by the factor shown in equation (20a). In Table II, the factors f' and f'' are computed from the bending stresses.⁷ They agree very well with the analytical error model predictions of equations (20a) and (20b).

In the second example, the additional stiffness parameters for the finite element results of an



$$E = 2 \times 10^7 \quad \nu = 0.2$$

$$W_{\text{FEM}} = 25.37$$

$$W_{\text{theory}} = 37.16$$

$$e_{\text{FEM}} = 0.46$$

$$e_{\text{[Eqn. 18a]}} = 0.46$$

Figure 5. Plane stress model of a beam undergoing flexure

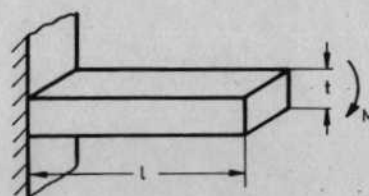


Figure 6. Deflection of a cantilever beam under loading using 8-node brick elements of displacement type

	$w(\text{fem})$	$e(\text{fem})$	$e(\text{equation (18a)})$
One element mesh	0.090	10.1	10.1
Two element mesh	0.278	2.6	2.6

idealization by $5 \times N$ 4-node elements,¹⁵ are computed (Figure 4). It agrees well with those predicted by equations (18a) and (18b). As the number of layers through the thickness is increased, the Poisson's effect is relieved and only the 'parasitic-shear' locking remains and this trend is predicted accurately by equation (18b).

Figure 5 reports the essential details of another example.¹⁶ It shows that an error of the second kind is present, which is accurately predicted by equation (18a).

Our final example extends equation (18a) to predict the behaviour of an exactly integrated displacement type 8-node brick element.¹⁷ The values of ν and l/t (Figure 6) are reworked from the values of deflections reported¹⁷ by assuming that an error of the second kind, as indicated by equation (18a), is present. The one-element and two-element models show a behaviour consistent with equation (18a) and the error model predictions agree with the finite element results.

EFFECTIVENESS OF REDUCED INTEGRATION, ETC.

It is instructive to examine how devices such as reduced integration, introduction of bubble-modes, etc., can improve the element.

We saw earlier that the shear strain field must be consistently modelled so that in the limiting situation the contributions to the shear energy can be separated into simple constraint terms which represent the true zero shear strain conditions. We saw from equations (4)–(6) that an exact integration of a shear strain field introduces 'spurious' constraints. A one-point integration of the shear energy term leaves only one term and this becomes the simple condition represented by equation (5a). The spurious constraint has thus been removed and the element does not lock.

We can also see how the addition of the incompatible modes improves the element.² The original field definitions are augmented by adding incompatible modes associated with additional internal variables. One way of doing this is

$$u = a_0 + a_1x + a_2y + a_3xy + a_4(1 - x^2) + a_5(1 - y^2)$$

$$v = b_0 + b_1x + b_2y + b_3xy + b_4(1 - x^2) + b_5(1 - y^2)$$

The shear strain field becomes

$$u_{,y} + v_{,x} = (a_2 + b_1) + (a_3 - 2b_4)x + (b_3 - 2a_5)y$$

Thus the field is now modelled consistently, in the sense that each of the coefficients, i.e. associated with constant, linear x , linear y , now comprise terms from both field functions. In the very thin limit, they give rise only to consistent constraints and hence there is no locking.

It is important that the additional modes must be chosen carefully to produce a consistent description of the shear strain. For example, element QM5 showed very poor behaviour when

exactly integrated.⁵ The displacement fields for this element were

$$u = a_0 + a_1x + a_2y + a_3xy + a_4(1-x^2)(1-y^2)$$

$$v = b_0 + b_1x + b_2y + b_3xy + b_4(1-x^2)(1-y^2)$$

The shear strain field then becomes

$$u_{,y} + v_{,x} = (a_2 + b_1) + (a_3 - 2b_4)x + (b_3 - 2a_4)y + (2a_4)yx^2 + (2b_4)xy^2$$

The coefficients associated with the yx^2 and xy^2 terms contain terms only from one interpolation function each. These can give rise to 'spurious' constraints. In fact, element QM⁵ could be used only after providing for reduced integration of the shear energy so that these 'spurious' constraints disappear.

CONCLUSIONS

This exercise unifies recent findings on the origin of errors of the second kind due to 'inconsistent' modelling of constrained strain fields in multi-field problems. It also provides a theoretical basis for explaining some of the 'tricks' that can salvage original elements afflicted by such errors. It can also give theoretical estimates for errors that occur when spurious constraining is present. The locking parameters thus identified may be of interest to mathematicians, so that the plane stress problem can be interpreted from the functional analysis point of view in a manner similar to that done for the Timoshenko beam¹⁸ and the curved beam¹⁹ recently.

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